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# An analytical tuning approach to multivariable model predictive controllers

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ABSTRACT

Multivariable model predictive control is a widely used advanced process control methodology, where handling delays and constraints are its key features. However, successful implementation of model predictive control requires an appropriate tuning of the controller parameters. This paper proposes an analytical tuning approach to multivariable model predictive controllers. The considered multivariable plants are square and consist of first-order plus dead time transfer functions. Most of the existing model predictive control tuning methods are based on trial and error or numerical approaches. In the case of no active constraints, closed loop transfer function matrices are derived and decoupling conditions are addressed. For control horizon of one, analytical tuning equations and achievable performances are obtained. Finally, simulation results are used to verify the effectiveness of the proposed tuning strategy.
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## 1. Introduction

Model predictive control (MPC) is employed in advanced process control strategies to control constrained multivariable processes and many successful implementations are reported [1–4].

In MPC design, model complexity leads to complicated closed loop plants which make analytical analysis difficult. This problem becomes more challenging for multivariable plants, where there are interactions between the loops. However, many industrial processes can be sufficiently described by first-order plus dead time (FOPDT) models. Therefore, in this paper FOPDT models are used to analyze the multivariable MPC in detail and closed loop studies lead to efficient tuning formulas. For square multi input-multi output FOPDT plants, two special cases are considered and studied in detail and analytical tuning equations are given to achieve desired performances. In addition, general square multivariable FOPDT models are also considered. In the first case, it is assumed that for each output the dynamics and time delay from the inputs are the same. In the second case, it is assumed that each input affects the outputs with the same dynamics and time delay. These two cases include many multivariable processes such as the pH neutralization process [5], some distillation processes [6] and mixing processes [7]. In the general case, closed loop studies are given and tuning equations are obtained.

Successful implementation of MPC in practical applications requires a proper tuning of its parameters which becomes more important and challenging for multivariable plants. MPC tuning parameters include the prediction and control horizons and the weight matrices used in the cost function. Tuning parameters are related to closed loop stability and desired performance characteristics in a complex and nonlinear manner and for multivariable plants it becomes extremely complicated, hence the MPC tuning problem is intricate and active constraints considerably complicates this problem. The issue of MPC tuning is addressed in many research papers [8,9]. However, the multivariable MPC tuning problem is considered in few articles. In addition, the available multivariable MPC tuning strategies are based on numerical methods or experimental and practical guidelines and there is only one analytical approach which gives closed form tuning equations [10]. In [10], an analytical tuning method is given for multivariable dynamic matrix control (DMC) parameters based on the multivariable FOPDT model approximation of the plant. The control effort weight matrix tuning is considered to avoid singularity in the control signal calculation, however closed loop performance is not considered. In [11], a multivariable MPC tuning method is proposed to achieve closed loop robust performance based on state estimation and sensitivity functions. In [12], a tuning method is introduced based

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on the constrained least square optimization that tunes multivariable MPC parameters to satisfy a predefined performance. In [13], by using a robust performance number a tuning procedure for MPC is developed which is applicable to multivariable non minimum phase plants. In [14], two methods for selecting the multivariable MPC weight matrices that result in linear state feedback controllers are derived and linear matrix inequality methods are used to numerically tune the weight matrices. A tuning method to achieve a desired closed loop performance is proposed in [15], where the weight matrices and the prediction horizon in multivariable MPC are tuned using a convex optimization approach. In [16] a tuning strategy is proposed for constrained multivariable MPC of uncertain plants based on particle swarm optimization techniques. In [17], an optimization based tuning procedure with certain robustness properties for an offset free MPC is presented for multivariable processes.

In [18], an analytical tuning strategy is developed for single input–single output MPC of FOPDT models when constraints are inactive, integral action augmentation is used to guarantee offset free step set point tracking and step disturbance rejection. Based on the results in [18], in this paper an analytical MPC tuning methodology is proposed for multivariable FOPDT models when the constraints are inactive. This extension to multivariable processes is achieved by modifying the cost function. This is necessary to simplify the closed loop equations and derivation of the tuning formulas. In this paper, closed loop transfer functions are derived based on the multivariable FOPDT model of the open loop plant. There are different formulations of MPC closed loop transfer functions [19]. However, the formulation derived in this paper matches the MPC closed loop transfer function to a desired closed loop transfer function. Employing this formulation, MPC tuning is restated as a pole placement problem and two key concepts namely the *achievability* and the *feasibility* characteristics are introduced. Also, closed loop decoupling conditions are derived with assumptions on the open loop structures. To derive exact tuning formulas with guaranteed closed loop stability and desired performance specifications, control horizons of one is considered and analytical tuning equations are obtained to achieve desired closed loop performance. Finally, simulation examples are used to show the effectiveness of the proposed tuning methodology. The paper is organized as follows. In Section 2, the problem formulation of MPC is given for multivariable FOPDT models. In Sections 3–5, the state space MPC formulation for two special cases of multivariable FOPDT plants and general case of multivariable FOPDT plants are given respectively and closed loop transfer functions are derived. Closed loop studies are provided and decoupling conditions are derived in the two special cases. Also, analytical MPC tuning equations are given for these cases. The efficiency of the proposed tuning algorithms is analyzed through examples in Section 6. Finally, Section 7 ends the paper with concluding the results.

## 2. Problem formulation

In this section, constrained MPC is formulated in the state-space framework for multivariable FOPDT models. Then, the multivariable MPC tuning fundamentals are given. Consider an $m$ inputs and $m$ outputs plant model as

$$\boldsymbol{y}(n) = \mathbf{G}(z)\boldsymbol{u}(n) \tag{1a}$$

where

$$\begin{aligned} &\boldsymbol{y}(n) = [y_1(n)\, y_2(n) \cdots y_m(n)]^T,\ \boldsymbol{u}(n) = [u_1(n)\, u_2(n) \cdots u_m(n)]^T \\ &\mathbf{G}(z) = [g_{ij}(z)],\ g_{ij}(z) = k_{ij}\frac{1-a_{ij}}{z-a_{ij}}z^{-d_{ij}} \quad i,j = 1,2,\ldots,m \end{aligned} \tag{1b}$$

and $y_i(n)$ is the $i$th output of the plant model and $u_j(n)$ is the $j$th control signal. Also $0 < a_{ij} < 1$ and $d_{ij}$ belongs to non negative integer sets of numbers for $i, j = 1, 2, \ldots, m$. Let, $\mathbf{K}$ be the steady state gain matrix of $\mathbf{G}(z)$ as follows

$$\mathbf{K} = [k_{ij}]\ i,j = 1,2,\ldots,m \tag{2}$$

Consider, the following state space realization of (1a) as

$$\begin{aligned} \boldsymbol{x}(n+1) &= \mathbf{A}\boldsymbol{x}(n) + \mathbf{B}\boldsymbol{u}(n) \\ \boldsymbol{y}(n) &= \mathbf{C}\boldsymbol{x}(n) \end{aligned} \tag{3}$$

Let the future model outputs be

$$\hat{\boldsymbol{y}}(n) = \mathbf{F}\boldsymbol{x}(n) + \mathbf{S}\bar{\boldsymbol{u}}(n) \tag{4}$$

where the matrices $\mathbf{F}$ and $\mathbf{S}$ are individually defined in each case as indicated in Sections 3–5, respectively. Also,

$$\begin{aligned} &\hat{\boldsymbol{y}}(n) = [\hat{\boldsymbol{y}}_1^T(n)\hat{\boldsymbol{y}}_2^T(n)\ldots\hat{\boldsymbol{y}}_m^T(n)]^T \\ &\hat{\boldsymbol{y}}_i(n) = [\hat{y}_i(n+\bar{d}_i+1|n)\hat{y}_i(n+\bar{d}_i+2|n)\ldots\hat{y}_i(n+\bar{d}_i+P_i|n)]^T\ i = 1,2,\ldots,m \\ &\bar{\boldsymbol{u}}(n) = [\boldsymbol{u}_1^T(n)\boldsymbol{u}_2^T(n)\ldots\boldsymbol{u}_m^T(n)]^T,\ \boldsymbol{u}_i(n) = [u_i(n)\, u_i(n+1)\ldots u_i(n+M_i-1)]^T\ i = 1,2,\ldots,m \end{aligned} \tag{5}$$

and $\hat{y}_i(\cdot|n)$ is the prediction of the $i$th model output at instance $n$, $P_i$ is the prediction horizon of the $i$th output, $M_i$ is the $i$th control horizon and $\bar{d}_i$ is the maximum delay between the $i$th output and input. The future plant outputs can be described as

$$\begin{aligned} &\hat{\boldsymbol{y}}_p(n) = [\hat{\boldsymbol{y}}_{p1}^T(n)\hat{\boldsymbol{y}}_{p2}^T(n)\ldots\hat{\boldsymbol{y}}_{pm}^T(n)]^T = \hat{\boldsymbol{y}}(n) + (\mathrm{diag}\{\mathbf{1}_{P_1\times 1},\ \mathbf{1}_{P_2\times 1},\ldots,\ \mathbf{1}_{P_m\times 1}\})\boldsymbol{b}(n) \\ &\hat{\boldsymbol{y}}_{pi}(n) = [\hat{y}_{pi}(n+\bar{d}_i+1|n)\hat{y}_{pi}(n+\bar{d}_i+2|n)\ldots\hat{y}_{pi}(n+\bar{d}_i+P_i|n)]^T,\ i = 1,2,\ldots,m \end{aligned} \tag{6}$$

where $\hat{y}_{pi}(\cdot|n)$ is the $i$th plant output prediction at instance $n$, $\mathbf{1} = [1\,1\ldots1]^T$ and $\boldsymbol{b}(n)$ is the bias term [1] which is calculated as

$$\boldsymbol{b}(n) = [b_1(n)\, b_2(n)\ldots b_m(n)]^T = \boldsymbol{y}_p(n) - \boldsymbol{y}(n),\ \boldsymbol{y}_p(n) = [y_{p1}(n)\, y_{p2}(n)\ldots y_{pm}(n)]^T \tag{7}$$

where $y_{pi}(n)$ is the $i$th plant output. The finite optimal control problem is defined as follows

$$\begin{aligned}&\min_{\bar{\boldsymbol{u}}(n)}(\bar{\boldsymbol{w}}(n)-\hat{\boldsymbol{y}}_p(n))^T\mathbf{Q}(\bar{\boldsymbol{w}}(n)-\hat{\boldsymbol{y}}_p(n))+(\boldsymbol{u}_{ss}(n)-\bar{\boldsymbol{u}}(n))^T\mathbf{R}(\boldsymbol{u}_{ss}(n)-\bar{\boldsymbol{u}}(n))\\&s.t.\ u_{i_{\min}}\le u_i(n+j)\le u_{i_{\max}}\quad j=0,1,\ldots,M_i-1,\ i=1,2,\ldots,m\\&y_{i_{\min}}\le\hat{y}_{pi}(n+j|n)\le y_{i_{\max}}\quad j=\bar{d}_i+1,\ \bar{d}_i+2,\ldots,\bar{d}_i+P_i,\ i=1,2,\ldots,m\end{aligned}\tag{8a}$$

where

$$\begin{aligned}&\bar{\boldsymbol{w}}(n)=(\mathrm{diag}\{\mathbf{1}_{P_1\times1},\ \mathbf{1}_{P_2\times1},\ldots,\mathbf{1}_{P_m\times1}\})\boldsymbol{w}(n),\ \ \boldsymbol{w}(n)=[w_1(n)w_2(n)\ldots w_m(n)]^T\\&\boldsymbol{u}_{ss}(n)=(\mathrm{diag}\{\mathbf{1}_{M_1\times1},\mathbf{1}_{M_2\times1},\ldots,\mathbf{1}_{M_m\times1}\})\mathbf{K}^{-1}(\boldsymbol{w}(n)-\boldsymbol{b}(n))\end{aligned}\tag{8b}$$

and $\mathbf{Q}$ is a positive semi definite matrix, $\mathbf{R}$ is a positive definite matrix and $w_i(.)$ is the $i$th reference signal. In (8a) $\boldsymbol{u}_{ss}(n)$ is the steady state value of the control signals. Note that the proposed method is not applicable for integrating processes.

In the case of no active constraints, the optimal control solution of (8a) is

$$\begin{aligned}\bar{\boldsymbol{u}}(n)=&(\mathbf{R}+\mathbf{S}^T\mathbf{Q}\mathbf{S})^{-1}(\mathbf{S}^T\mathbf{Q}(\mathrm{diag}\{\mathbf{1}_{P_1\times1},\mathbf{1}_{P_2\times1},\ldots,\mathbf{1}_{P_m\times1}\})[\boldsymbol{w}(n)-\boldsymbol{b}(n)]-\mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}}\bar{\boldsymbol{x}}(n)\\&+\mathbf{R}\mathbf{K}^{-1}(\mathrm{diag}\{\mathbf{1}_{M_1\times1},\mathbf{1}_{M_2\times1},\ldots,\mathbf{1}_{M_m\times1}\})[\boldsymbol{w}(n)-\boldsymbol{b}(n)])\end{aligned}\tag{9}$$

where $\bar{\mathbf{F}}$ and $\bar{\boldsymbol{x}}(n)$ are derived from $\mathbf{F}$ and $\boldsymbol{x}(n)$, respectively. Due to the state space realization of (1a), some zero columns appear in $\mathbf{F}$. To simplify the vector multiplications in (4), matrix $\mathbf{F}$ is formed from $\mathbf{F}$ by removing its zero columns and $\bar{\boldsymbol{x}}(n)$ represents the corresponding states in $\boldsymbol{x}(n)$. Let

$$\begin{aligned}&\bar{\boldsymbol{\Psi}}=(\mathbf{R}+\mathbf{S}^T\mathbf{Q}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}}\\&\bar{\boldsymbol{\Xi}}=(\mathbf{R}+\mathbf{S}^T\mathbf{Q}\mathbf{S})^{-1}(\mathbf{S}^T\mathbf{Q}(\mathrm{diag}\{\mathbf{1}_{P_1\times1},\mathbf{1}_{P_2\times1},\ldots,\mathbf{1}_{P_m\times1}\})+\mathbf{R}(\mathrm{diag}\{\mathbf{1}_{M_1\times1},\mathbf{1}_{M_2\times1},\ldots,\mathbf{1}_{M_m\times1}\})\mathbf{K}^{-1})\end{aligned}\tag{10}$$

Hence, Eq. (9) leads to

$$\bar{\boldsymbol{u}}(n)=\bar{\boldsymbol{\Xi}}(\boldsymbol{w}(n)-\boldsymbol{b}(n))-\bar{\boldsymbol{\Psi}}\bar{\boldsymbol{x}}(n)\tag{11}$$

Now, the current control signals which are applied to the system are

$$\boldsymbol{u}(n)=\boldsymbol{\Xi}(\boldsymbol{w}(n)-\boldsymbol{b}(n))-\boldsymbol{\Psi}\bar{\boldsymbol{x}}(n)\tag{12a}$$

where $\boldsymbol{\Xi}$ and $\boldsymbol{\Psi}$ are chosen properly from the elements of $\bar{\boldsymbol{\Xi}}$ and $\bar{\boldsymbol{\Psi}}$. To derive $\boldsymbol{\Xi}$ from the elements of $\bar{\boldsymbol{\Xi}}$ we use the following selection matrix $\mathbf{X}$

$$\begin{aligned}&\Xi=\mathbf{X}\bar{\boldsymbol{\Xi}}\\&\mathbf{X}=\begin{bmatrix}\mathbf{X}_1&\mathbf{X}_2&\cdots&\mathbf{X}_m\end{bmatrix},\ \mathbf{X}_i=\begin{bmatrix}&&\mathbf{0}_{(i-1)\times M_i}&&\\1&0&0&\ldots&0\\&&\mathbf{0}_{(m-i)\times M_i}&&\end{bmatrix}_{m\times M_i},\quad i=1,2,\ldots,m\end{aligned}\tag{12b}$$

A similar selection matrix can be used to derive $\boldsymbol{\Psi}$ from $\bar{\boldsymbol{\Psi}}$. Hence, (12a) indicates that the desired control signals can be derived by selecting the desired gain matrices $\boldsymbol{\Psi}_d$ and $\boldsymbol{\Xi}_d$. And the tuning problem can be reformulated to obtain analytical relations between these desired gain matrices and $\mathbf{Q}$, $\mathbf{R}$, $P_i$, $M_j$ for $i,j=1,2,\ldots,m$ as the tuning parameters. Also, in this framework *achievable* performances for the MPC of multivariable FOPDT models are applicable. However, it is important to note that not any desired performance is *achievable*. In addition, some achievable gain matrices do not ensure closed loop stability and are therefore not *feasible*. In what follows, the key *achievability* and *feasibility* concepts are defined. It will be shown that, $\boldsymbol{\Xi}$ is dependent on $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}$ is selected as the tuning parameter.

**Definition 1.** The desired gain matrix $\boldsymbol{\Psi}_d$ that satisfies Eq. (10) is called the *achievable gain matrix.*

**Definition 2.** The achievable gain matrix $\boldsymbol{\Psi}_d$ that ensures closed loop stability is called the *feasible gain matrix.*

## 3. MPC formulation and tuning equations for multivariable FOPDT models: case 1

In this section, constrained MPC is formulated in the state-space framework for a special class of multivariable FOPDT models. In this case, it is assumed that for each output the dynamics and time delay from the inputs are the same. In the case of no active constraints and no model mismatch, the closed loop behavior of this structure is studied in detail. It is shown that closed loop decoupling is possible by appropriately tuning the controller parameters and the decoupling conditions are derived. Finally, tuning formulations are given to achieve the desired closed loop performance and also achievable performances are addressed.

### 3.1. MPC formulation for MIMO FOPDT models

Consider the transfer function of model (1b) as

$$g_{ij}(z)=k_{ij}\frac{1-a_i}{z-a_i}z^{-d_i}\quad i,j=1,2,\ldots,m\tag{13a}$$

Let, $\mathbf{P}(z)$ and $\mathbf{D}(z)$ be the dynamic and delay matrices of (1b) and (13a) as

$$\mathbf{P}(z) = \mathrm{diag}\left\{\frac{1-a_1}{z-a_1}, \frac{1-a_2}{z-a_2}, \ldots, \frac{1-a_m}{z-a_m}\right\}, \quad \mathbf{D}(z) = \mathrm{diag}\{z^{-d_1}, z^{-d_2}, \ldots, z^{-d_m}\} \tag{13b}$$

therefore,

$$\mathbf{G}(z) = \mathbf{P}(z)\mathbf{D}(z)\mathbf{K} \tag{13c}$$

The state space realization between $y_i(n)$ for $i = 1,2,\ldots,m$ and the inputs can be presented as

$$\begin{aligned} \boldsymbol{x}_i(n+1) &= \mathbf{A}_i\boldsymbol{x}_i(n) + \mathbf{B}_i\boldsymbol{u}(n) \\ y_i(n) &= \mathbf{C}_i\boldsymbol{x}_i(n) \end{aligned} \tag{14a}$$

where

$$\begin{aligned} &\boldsymbol{x}_i(n) = [y_i(n)y_i(n+1)\ldots y_i(n+d_i)]^T \\ &\mathbf{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & a_i \end{bmatrix}, \ \mathbf{B}_i = (1-a_i)\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & 0 \\ k_{i1} & k_{i2} & \cdots & k_{im} \end{bmatrix} \\ &\mathbf{C}_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned} \tag{14b}$$

and the state space realization of (13c) is as Eq. (3) with

$$\begin{aligned} &\boldsymbol{x}(n) = [\boldsymbol{x}_1(n)^T\, \boldsymbol{x}_2(n)^T \ldots \boldsymbol{x}_m(n)^T]^T, \ \mathbf{A} = \mathrm{diag}\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m\} \\ &\mathbf{B} = [\mathbf{B}_1^T\, \mathbf{B}_2^T \ldots \mathbf{B}_m^T]^T, \ \mathbf{C} = \mathrm{diag}\{\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_m\} \end{aligned} \tag{15}$$

The above realization is both controllable and observable. Now, due to (4) the future model output values can be calculated with

$$\mathbf{F} = \mathrm{diag}\{\mathbf{F}_{11}, \mathbf{F}_{22}, \ldots, \mathbf{F}_{mm}\}, \mathbf{F}_{ii} = \begin{bmatrix} \mathbf{C}_i\mathbf{A}_i^{d_i+1} \\ \mathbf{C}_i\mathbf{A}_i^{d_i+2} \\ \vdots \\ \mathbf{C}_i\mathbf{A}_i^{d_i+P_i} \end{bmatrix} = [\,\mathbf{0} \quad \mathbf{0} \quad \ldots \quad \mathbf{0} \quad \bar{\mathbf{F}}_{ii}\,], \bar{\mathbf{F}}_{ii} = \begin{bmatrix} a_i \\ a_i^2 \\ \vdots \\ a_i^{P_i} \end{bmatrix} \ i = 1, 2, \ldots, m$$

where $\mathbf{0}$ is a zero matrix with proper dimension.

$$\bar{\mathbf{F}} = \mathrm{diag}\{\bar{\mathbf{F}}_{11}, \bar{\mathbf{F}}_{22}, \ldots, \bar{\mathbf{F}}_{mm}\}, \quad \mathbf{S} = [\mathbf{S}_{ij}] \quad i, j = 1, 2, \ldots, m$$

$$\mathbf{S}_{ij} = k_{ij}(1-a_i)\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_i & 1 & \ddots & \vdots & \vdots \\ \vdots & a_i & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ \vdots & \vdots & \cdots & a_i & 1 \\ \vdots & \vdots & \cdots & \vdots & 1+a_i \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_i^{P_i-1} & a_i^{P_i-2} & \cdots & a_i^{P_i-M_j+1} & 1+a_i+\cdots+a_i^{P_i-M_j} \end{bmatrix} \ i, j = 1, 2, \ldots, m \tag{16}$$

Note that in this case we have $\bar{d}_i = d_i$ for $i = 1, 2, \ldots, m$. Some mathematical manipulations on the optimal solution (12a) lead to

$$\boldsymbol{u}(n) = \boldsymbol{\Xi}(\boldsymbol{w}(n) - \boldsymbol{b}(n)) - \boldsymbol{\Psi}\,\mathbf{D}(z)^{-1}\boldsymbol{y}(n) \tag{17}$$

where

$$\boldsymbol{\Xi} = [\xi_{ij}], \ \boldsymbol{\Psi} = [\psi_{ij}] \quad i, j = 1, 2, \ldots, m \tag{18}$$

$\Xi$ and $\Psi$ are appropriately selected from the elements of $\bar{\Xi}$ and $\bar{\Psi}$ in (10) from (12b) with $\mathbf{S}$ and $\bar{\mathbf{F}}$ as defined in (16). In (17), due to $\mathbf{D}(z)^{-1}\boldsymbol{y}(n)$, the predicted values $y_i(n+d_i)$, $i=1,2,\ldots,m$ are needed. Mathematical operations show that

$$y_i(n+d_i) = a_i^{d_i} y_i(n) + (1-a_i)\left(\sum_{s=1}^{d_i} a_i^{s-1}\left(\sum_{j=1}^{m} k_{ij}u_j(n-d_i+s-1)\right)\right) \quad i=1,2,\ldots,m \tag{19}$$

Using (17) and (19), the control signals are calculated.

**Closed loop analysis**. Let $\boldsymbol{b}(n)=\mathbf{0}$, that is the plant and model outputs are the same. Using (7), (13c) and (17), we have

$$(\mathbf{I}+\mathbf{P}(z)\mathbf{D}(z)\mathbf{K}\,\boldsymbol{\Psi}\,\mathbf{D}(z)^{-1})\boldsymbol{y}_p(n) = \mathbf{P}(z)\mathbf{D}(z)\mathbf{K}\,\Xi\,\boldsymbol{w}(n) \tag{20}$$

It can be shown that

$$\Xi = \mathbf{K}^{-1} + \boldsymbol{\Psi} \tag{21}$$

Define the loop gain matrix $\mathbf{L}$ as

$$\mathbf{L} = \mathbf{K}\,\boldsymbol{\Psi} = [l_{ij}] \quad i,j=1,2,\ldots,m \tag{22}$$

Mathematical manipulations on (20)–(22) lead to the closed loop relation as

$$\boldsymbol{y}_p(n) = \mathbf{G}_{cl}(z)\boldsymbol{w}(n) \tag{23a}$$

where

$$\mathbf{G}_{cl}(z) = \mathbf{D}(z)[\mathbf{P}(z)^{-1}+\mathbf{L}]^{-1}(\mathbf{I}+\mathbf{L}) \tag{23b}$$

**Theorem 1.** In the case of no active constraint and no model mismatch, the optimization problem (8a) for the multivariable FOPDT transfer function (13c) leads to the following decoupled desired closed loop transfer function matrix

$$\mathbf{G}_{cl_d}(z) = \mathrm{diag}\left\{\frac{1-x_1}{z^{d_1}(z-x_1)}, \frac{1-x_2}{z^{d_2}(z-x_2)}, \ldots, \frac{1-x_m}{z^{d_m}(z-x_m)}\right\} \tag{24a}$$

if the following loop gain matrix is chosen

$$\mathbf{L} = \mathrm{diag}\{l_{11}, l_{22}, \ldots, l_{mm}\},\ l_{ii} = \frac{a_i - x_i}{1-a_i} \quad i=1,2,\ldots,m \tag{24b}$$

**Proof:** Follows from (23b) and (24a).

Also, partial decoupling is achievable.

**Partial decoupling: Case 1.** Consider $l_{ij}=0$ for $i,j=1,2,\ldots,m$, $i\neq j$, $i\neq h$. In this case, all outputs are decoupled except for the $h$th output. That is,

$$\begin{aligned}
&\mathbf{G}_{cl}(z) = [g_{cl_{ij}}(z)] \quad i,j=1,2,\ldots,m\\
&g_{cl_{ii}}(z) = \frac{(1-a_i)(1+l_{ii})}{z^{d_i}[z-a_i+l_{ii}(1-a_i)]} \quad i=1,2,\ldots,m,\ g_{cl_{ij}}(z)=0\ i,j=1,2,\ldots,m,\ i\neq j,\ i\neq h\\
&g_{cl_{hj}}(z) = \frac{l_{hj}(1-a_h)(z-1)}{z^{d_h}[z-a_h+l_{hh}(1-a_h)][z-a_j+l_{jj}(1-a_j)]}\ j=1,2,\ldots,m
\end{aligned} \tag{25}$$

**Partial decoupling: Case 2.** Let in the $h$th row of $\mathbf{L}$, $l_{hj}=0$ for $j=1,2,\ldots,m, j\neq h$ and $l_{ij}\neq 0$ for $i,j=1,2,\ldots,m, i\neq h$. In this case, only the $h$th output is decoupled and we have

$$y_h(n) = \frac{(1-a_h)(1+l_{hh})}{z^{d_h}[z-a_h+l_{hh}(1-a_h)]}u_h(n) \tag{26}$$

### 3.2. Tuning formulas for the desired performance

In this section, assuming control horizon of one, the tuning formulas for the given model predictive controller are derived when the constraints are inactive.

Using the closed loop transfer function matrix (23b), it is possible to shape the closed loop transfer function in a desired form by appropriately choosing the desired gain matrix $\boldsymbol{\Psi}_d$. Note that (22) gives the desired loop gain matrix, $\mathbf{L}_d=\mathbf{K}\,\boldsymbol{\Psi}_d$. Consider the control horizon of one in Eqs. (10) and (18). This leads to

$$\boldsymbol{\Psi} = (\mathbf{R}+\mathbf{S}^T\boldsymbol{Q}\,\mathbf{S})^{-1}\mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}} \tag{27}$$

from (16), we have

$$\mathbf{S} = \bar{\mathbf{S}}\,\mathbf{K} = (\mathrm{diag}\{\bar{\mathbf{S}}_{11}, \bar{\mathbf{S}}_{22}, \ldots, \bar{\mathbf{S}}_{mm}\})\mathbf{K},\ \bar{\mathbf{S}}_{ii} = [1-a_i\ 1-a_i^2\ldots 1-a_i^{P_i}]^T \quad i=1,2,\ldots,m \tag{28}$$

**Theorem 2.** In the case of no active constraint and no model mismatch, the optimization problem (8a) for the multivariable FOPDT model (13c) leads to the closed loop transfer function (23b) with the desired achievable loop gain matrix $\mathbf{L}_d$ that satisfies the following inequality

$$\mathbf{L}_d < (\bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{S}})^{-1}\bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{F}} \tag{29}$$

and the feasibility conditions of this desired loop gain matrix is the intersection of (29) and the stability conditions of the closed loop plant (23a). Selecting $\mathbf{R}$ as the tuning parameter, the tuning equation for achieving this desired loop gain matrix is

$$\mathbf{R} = \mathbf{K}^T\bar{\mathbf{S}}^T\mathbf{Q}[\bar{\mathbf{F}}\mathbf{L}_d^{-1} - \bar{\mathbf{S}}]\mathbf{K} \tag{30}$$

**Proof:** According to Eq. (28), we have

$$\mathbf{S}^T\mathbf{Q}\mathbf{S} = \mathbf{K}^T\bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{S}}\mathbf{K} \tag{31a}$$

using Eqs. (27) and (31a), $\mathbf{R}$ is normalized as

$$\mathbf{R} = \mathbf{K}^T\bar{\mathbf{R}}\mathbf{K} \tag{31b}$$

Hence, (31a), (31b) and (27), give

$$\mathbf{K}^T(\bar{\mathbf{R}} + \bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{S}})\mathbf{K}\,\boldsymbol{\Psi} = \mathbf{K}^T\bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{F}} \rightarrow (\bar{\mathbf{R}} + \bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{S}}) = \bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{F}}\mathbf{L}^{-1} \rightarrow \bar{\mathbf{R}} = \bar{\mathbf{S}}^T\mathbf{Q}[\bar{\mathbf{F}}\mathbf{L}^{-1} - \bar{\mathbf{S}}] \tag{32}$$

And the assertion follows from (32) and (31b). Note that $\mathbf{R}$ should be a positive definite matrix, so

$$\mathbf{K}^T\bar{\mathbf{S}}^T\mathbf{Q}[\bar{\mathbf{F}}\mathbf{L}^{-1} - \bar{\mathbf{S}}]\mathbf{K} > 0 \tag{33}$$

and the invertibility of $\mathbf{K}$ gives,

$$\bar{\mathbf{S}}^T\mathbf{Q}[\bar{\mathbf{F}}\mathbf{L}^{-1} - \bar{\mathbf{S}}] > 0 \rightarrow \bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{F}} - \bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{S}}\mathbf{L} > 0 \rightarrow \mathbf{L} < (\bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{S}})^{-1}\bar{\mathbf{S}}^T\mathbf{Q}\bar{\mathbf{F}}$$

and the proof is completed.

## 4. MPC formulation and tuning strategy for multivariable FOPDT models: case 2

In this section, constrained MPC is formulated in the state-space framework for another special class of multivariable FOPDT models. In this case, it is assumed that each input affects the outputs with the same dynamics and time delay. In the case of no active constraint and no model mismatch, the closed loop behavior of this structure is studied in detail. In a special case, the decoupling conditions of the closed loop plant are derived. Finally, tuning formulations are given to achieve the desired closed loop performance and also achievable performances are addressed.

### 4.1. State space MPC formulation for MIMO FOPDT models

For case 2, the transfer function of model (1b) is

$$g_{ij}(z) = k_{ij}\frac{1-a_j}{z-a_j}z^{-d_j} \quad i,j = 1,2,\ldots,m \tag{34a}$$

Using (2), (13b) and (34a), the transfer function model of the second case can be rewritten as

$$\mathbf{G}(z) = \mathbf{K}\mathbf{P}(z)\mathbf{D}(z) \tag{34b}$$

For the $i$th output, we have

$$y_i(n) = \sum_{j=1}^{m} k_{ij}v_j(n) \quad i = 1,2,\ldots,m \tag{35a}$$

where

$$v_i(n) = \frac{1-a_i}{z-a_i}u_i(n-d_i) \quad i = 1,2,\ldots,m \tag{35b}$$

The state space realization between each $\boldsymbol{v}_i(n)$ for $i = 1, 2, \ldots, m$ and plant inputs can be written as

$$\begin{aligned} \boldsymbol{x}_i(n+1) &= \mathbf{A}_i\boldsymbol{x}_i(n) + \mathbf{B}_i\boldsymbol{u}(n) \\ v_i(n) &= \mathbf{C}_i\boldsymbol{x}_i(n) \end{aligned} \tag{36a}$$

where

$$\boldsymbol{x}_i(n) = \begin{bmatrix} v_i(n) & v_i(n+1) & \cdots & v_i(n+d_i) \end{bmatrix}^T$$

$$\mathbf{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & a_i \end{bmatrix}, \quad \mathbf{B}_i = (1-a_i)\begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \underset{i\text{th}}{1} & 0 & \cdots & 0 \end{bmatrix} \tag{36b}$$

$$\mathbf{C}_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence, the state space realization of the transfer function matrix (34b) is as Eq. (3) with

$$\begin{aligned}&\boldsymbol{x}(n)=[\boldsymbol{x}_1(n)^T\,\boldsymbol{x}_2(n)^T\ldots\boldsymbol{x}_m(n)^T]^T,\ \mathbf{A}=\mathrm{diag}\{\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{A}_m\}\\&\mathbf{B}=[\mathbf{B}_1^T\,\mathbf{B}_2^T\ldots\mathbf{B}_m^T]^T,\mathbf{C}=\mathbf{K}\,\mathrm{diag}\{\mathbf{C}_1,\mathbf{C}_2,\ldots,\mathbf{C}_m\}\end{aligned}\tag{37}$$

The above state space realization is minimal. Now, from (4) the future model output values can be calculated with

$$\begin{aligned}&\mathbf{F}=[\mathbf{F}_{ij}],\ \ \mathbf{S}=[\mathbf{S}_{ij}]\,i,j=1,2,\ldots,m\\&\mathbf{F}_{ij}=k_{ij}\begin{bmatrix}\mathbf{C}_j\mathbf{A}_j^{d_j+1}\\\mathbf{C}_j\mathbf{A}_j^{d_j+2}\\\vdots\\\mathbf{C}_j\mathbf{A}_j^{d_j+P_i}\end{bmatrix}=[\,\mathbf{0}\quad\mathbf{0}\quad\cdots\quad\mathbf{0}\quad\bar{\mathbf{F}}_{ij}\,],\ \bar{\mathbf{F}}_{ij}=k_{ij}a_j^{(\bar{d}-d_j)}\begin{bmatrix}a_j\\a_j^2\\\vdots\\a_j^{P_i}\end{bmatrix},\bar{\mathbf{F}}=[\bar{\mathbf{F}}_{ij}]\quad i,j=1,2,\ldots,m\\&\tilde{\mathbf{S}}_{ij}=k_{ij}(1-a_j)\begin{bmatrix}a_j^{\bar{d}-d_j}&a_j^{\bar{d}-d_j-1}&\cdots&a_j^{\bar{d}-d_j-M_j+2}&\sum_{s=1}^{1}a_j^{\bar{d}-d_j+s-M_j}\\a_j^{\bar{d}-d_j+1}&a_j^{\bar{d}-d_j}&\cdots&a_j^{\bar{d}-d_j-M_j+3}&\sum_{s=1}^{2}a_j^{\bar{d}-d_j+s-M_j}\\\vdots&\vdots&&\vdots&\vdots\\a_j^{\bar{d}-d_j+P_i-1}&a_j^{\bar{d}-d_j+P_i-2}&\cdots&a_j^{\bar{d}-d_j+P_i-M_j+1}&\sum_{s=1}^{P_i}a_j^{\bar{d}-d_j+s-M_j}\end{bmatrix}\quad i,j=1,2,\ldots,m\end{aligned}\tag{38}$$

Note that, $a_j^x=\begin{cases}0 & x<0\\ a_j^x & x\geq 0\end{cases}$ for $j=1,2,\ldots,m$ and in this case we have $\bar{d}_i=\bar{d}=\max_{j=1,2,\ldots,m}\{d_j\}$ for $i=1,2,\ldots,m$. Hence, the optimal solution is the same as (12a)

$$\boldsymbol{u}(n)=\boldsymbol{\Xi}(\boldsymbol{w}(n)-\boldsymbol{b}(n))-\boldsymbol{\Psi}\,\tilde{\boldsymbol{x}}(n)\tag{39}$$

where

$$\begin{aligned}&\boldsymbol{\Xi}=[\xi_{ij}],\ \ \boldsymbol{\Psi}=[\psi_{ij}]\quad i,j=1,2,\ldots,m\\&\tilde{\boldsymbol{x}}(n)=[v_1(n+d_1)\,v_2(n+d_2)\ldots v_m(n+d_m)]^T\end{aligned}\tag{40}$$

where $\boldsymbol{\Xi}$ and $\boldsymbol{\Psi}$ are chosen properly from the elements of $\bar{\boldsymbol{\Xi}}$ and $\bar{\boldsymbol{\Psi}}$ in (10) from (12b). The predicted values of $v_i(n+d_i),\ i=1,2,\ldots,m$ are required in (39) and (40). These are derived by mathematical manipulations as

$$v_i(n+d_i)=a_i^{d_i}v_i(n)+(1-a_i)\sum_{s=1}^{d_i}a_i^{s-1}u_i(n-d_i+s-1)\quad i=1,2,\ldots,m\tag{41}$$

**Closed loop analysis.** Let $\boldsymbol{b}(n)=\mathbf{0}$, that is the plant and model outputs are the same. Using (34b) and (39), we have

$$[\mathbf{P}(z)^{-1}+\boldsymbol{\Psi}]\mathbf{D}(z)^{-1}\mathbf{K}^{-1}\boldsymbol{y}_p(n)=\boldsymbol{\Xi}\,\boldsymbol{w}(n)\tag{42}$$

It can be shown that

$$\boldsymbol{\Xi}=(\mathbf{I}+\boldsymbol{\Psi})\mathbf{K}^{-1}\tag{43}$$

Hence, the closed loop relation can be obtained as

$$\boldsymbol{y}_p(n)=\mathbf{G}_{cl}(z)\boldsymbol{w}(n)\tag{44a}$$

where

$$\mathbf{G}_{cl}(z)=\mathbf{K}\,\mathbf{D}(z)[\mathbf{P}(z)^{-1}+\boldsymbol{\Psi}]^{-1}(\mathbf{I}+\boldsymbol{\Psi})\mathbf{K}^{-1}\tag{44b}$$

In general, it is not possible to decouple the closed loop plant. However, in the case of $d_1=d_2=\ldots=d_m$ this is achievable, as stated in the following lemma.

**Lemma 1.** Suppose $d=d_i$ for $i=1,2,\ldots,m$. In the case of no active constraint and no model mismatch, the optimization problem (8a) for the multivariable FOPDT model (34b) leads to the following decoupled desired closed loop transfer function matrix

$$\mathbf{G}_{cl_d}(z)=\mathrm{diag}\left\{\frac{1-x_1}{z^d[z-x_1]},\ \frac{1-x_2}{z^d[z-x_2]},\ldots,\frac{1-x_m}{z^d[z-x_m]}\right\}\tag{45a}$$

if the following gain matrix is chosen

$$\boldsymbol{\Psi}=(\mathbf{K}^{-1}[z^d\mathbf{G}_{cl_d}(z)]^{-1}\mathbf{K}-\mathbf{P}(z)^{-1})(\mathbf{I}-\mathbf{K}^{-1}[z^d\mathbf{G}_{cl_d}(z)]^{-1}\mathbf{K})^{-1}\tag{45b}$$

### 4.2. Tuning formulas for desired performance

In this section, assuming control horizon of one, the tuning formulas for the given model predictive controller are derived when the constraints are inactive.

**Theorem 3.** In the case of no active constraint and no model mismatch, the optimization problem (8a) for the multivariable FOPDT model (34b) leads to the closed loop transfer function matrix (44b) with the desired achievable gain matrix $\boldsymbol{\Psi}_d$ that satisfies the following inequality

$$\boldsymbol{\Psi}_d < (\mathbf{S}^T\mathbf{Q}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}} \tag{46}$$

and the feasibility conditions of this desired gain matrix is the intersection of (46) and the stability conditions of the closed loop plant (44a). By selecting $\mathbf{R}$ as the tuning parameter, the tuning equation for achieving this desired gain matrix is

$$\mathbf{R} = \mathbf{S}^T\mathbf{Q}[\bar{\mathbf{F}}\boldsymbol{\Psi}_d^{-1} - \mathbf{S}] \tag{47}$$

**Proof:** The proof is similar to that of Theorem 2.

## 5. MPC formulation and tuning strategy for multivariable FOPDT models: the general case

In this section, constrained MPC is formulated in the state-space framework for the general multivariable FOPDT models, i.e. (1b). Then, the closed loop transfer function is obtained.

### 5.1. State space MPC formulation for MIMO FOPDT models

Consider the general transfer function matrix given by (1b). Let $\bar{\mathbf{G}}(z)$ describes the non-delay part of the transfer function matrix

$$\bar{\mathbf{G}}(z) = [\bar{g}_{ij}(z)], \quad \bar{g}_{ij}(z) = k_{ij}\frac{1-a_{ij}}{z-a_{ij}} \quad i,j = 1,2,\ldots,m \tag{48}$$

The $i$th output of (48) is

$$y_i(n) = \sum_{j=1}^{m} v_{ij}(n) \quad i = 1,2,\ldots,m \tag{49a}$$

where

$$v_{ij}(n) = \bar{g}_{ij}(z)u_j(n-d_{ij}) \quad i,j = 1,2,\ldots,m \tag{49b}$$

and the corresponding state space realization can be written as

$$\begin{aligned} \boldsymbol{x}_{ij}(n+1) &= \mathbf{A}_{ij}\boldsymbol{x}_{ij}(n) + \mathbf{B}_{ij}\boldsymbol{u}(n) \\ v_{ij}(n) &= \mathbf{C}_{ij}\boldsymbol{x}_{ij}(n) \end{aligned} \tag{50a}$$

where

$$\begin{aligned} \boldsymbol{x}_{ij}(n) &= \begin{bmatrix} v_{ij}(n) & v_{ij}(n+1) & \cdots & v_{ij}(n+d_{ij}) \end{bmatrix}^T \\ \mathbf{A}_{ij} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & a_{ij} \end{bmatrix}, \quad \mathbf{B}_{ij} = k_{ij}(1-a_{ij}) \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \underset{j\text{th}}{1} & 0 & \cdots & 0 \end{bmatrix} \\ \mathbf{C}_{ij} &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned} \tag{50b}$$

Hence, the state space realization of the transfer function matrix (1b) is as (3) with

$$\begin{aligned} \boldsymbol{x}(n) &= [\,\boldsymbol{x}_{11}(n)^T \;\cdots\; \boldsymbol{x}_{1m}(n)^T \;\; \boldsymbol{x}_{21}(n)^T \;\cdots\; \boldsymbol{x}_{2m}(n)^T \;\cdots\; \boldsymbol{x}_{m1}(n)^T \;\cdots\; \boldsymbol{x}_{mm}(n)^T\,]^T \\ \mathbf{A} &= \mathrm{diag}\{\mathbf{A}_{11}\cdots\mathbf{A}_{1m}\,\mathbf{A}_{21}\cdots\mathbf{A}_{2m}\cdots\mathbf{A}_{m1}\cdots\mathbf{A}_{mm}\} \\ \mathbf{B} &= \left[\mathbf{B}_{11}^T\cdots\mathbf{B}_{1m}^T\,\mathbf{B}_{21}^T\cdots\mathbf{B}_{2m}^T\cdots\mathbf{B}_{m1}^T\cdots\mathbf{B}_{mm}^T\right]^T \\ \mathbf{C} &= \begin{bmatrix} \mathbf{C}_{11} & \cdots & \mathbf{C}_{1m} & & & & & & \\ & & & \mathbf{C}_{21} & \cdots & \mathbf{C}_{2m} & & 0 & \\ & 0 & & & & & \ddots & & \\ & & & & & & \mathbf{C}_{m1} & \cdots & \mathbf{C}_{mm} \end{bmatrix} \end{aligned} \tag{51}$$

The above non minimal realization facilitates the closed loop analysis and tuning procedure. Now, the future model output values can be calculated with

$$\mathbf{F}=\begin{bmatrix}\mathbf{F}_{11} & \cdots & \mathbf{F}_{1m} & & & & & \\ & & & \mathbf{F}_{21} & \cdots & \mathbf{F}_{2m} & & \mathbf{0} \\ & \mathbf{0} & & & & & \ddots & \\ & & & & & & \mathbf{F}_{m1} \cdots & \mathbf{F}_{mm}\end{bmatrix}$$

$$\mathbf{F}_{ij}=\begin{bmatrix}\mathbf{C}_{ij}\mathbf{A}_{ij}^{d_{ij}+1}\\ \mathbf{C}_{ij}\mathbf{A}_{ij}^{d_{ij}+2}\\ \vdots\\ \mathbf{C}_{ij}\mathbf{A}_{ij}^{d_{ij}+P_i}\end{bmatrix}=[\,\mathbf{0}\quad\mathbf{0}\quad\cdots\quad\mathbf{0}\quad\bar{\mathbf{F}}_{ij}\,],\quad \bar{\mathbf{F}}_{ij}=a_{ij}^{(\bar{d}_i-d_{ij})}\begin{bmatrix}a_{ij}\\ a_{ij}^2\\ \vdots\\ a_{ij}^{P_i}\end{bmatrix},\quad \bar{\mathbf{F}}=[\bar{\mathbf{F}}_1\,\bar{\mathbf{F}}_2\cdots\bar{\mathbf{F}}_m]\quad i,j=1,2,\ldots,m$$

$$\bar{\mathbf{F}}_1=\begin{bmatrix}\bar{\mathbf{F}}_{11} & \cdots & \bar{\mathbf{F}}_{1m}\\ & & \\ & \mathbf{0} & \end{bmatrix},\quad \bar{\mathbf{F}}_2=\begin{bmatrix}& \mathbf{0} & \\ \bar{\mathbf{F}}_{21} & \cdots & \bar{\mathbf{F}}_{2m}\\ & \mathbf{0} & \end{bmatrix},\quad \cdots,\quad \bar{\mathbf{F}}_m=\begin{bmatrix}& \mathbf{0} & \\ & & \\ \bar{\mathbf{F}}_{m1} & \cdots & \bar{\mathbf{F}}_{m1}\end{bmatrix} \tag{52}$$

$$\mathbf{S}=[\mathbf{S}_{ij}]\quad i,j=1,2,\ldots,m$$

$$\mathbf{S}_{ij}=k_{ij}(1-a_{ij})\begin{bmatrix}a_{ij}^{\bar{d}_i-d_{ij}} & a_{ij}^{\bar{d}_i-d_{ij}-1} & \cdots & a_{ij}^{\bar{d}_i-d_{ij}-M_j+2} & \sum_{s=1}^{1}a_{ij}^{\bar{d}_i-d_{ij}+s-M_j}\\ a_{ij}^{\bar{d}_i-d_{ij}+1} & a_{ij}^{\bar{d}-d_{ij}} & \cdots & a_{ij}^{\bar{d}_i-d_{ij}-M_j+3} & \sum_{s=1}^{2}a_{ij}^{\bar{d}_i-d_{ij}+s-M_j}\\ \vdots & \vdots & & \vdots & \vdots\\ a_{ij}^{\bar{d}_i-d_{ij}+P_i-1} & a_{ij}^{\bar{d}_i-d_{ij}+P_i-2} & \cdots & a_{ij}^{\bar{d}_i-d_{ij}+P_i-M_j+1} & \sum_{s=1}^{P_i}a_{ij}^{\bar{d}_i-d_{ij}+s-M_j}\end{bmatrix}$$

Note that, $a_{ij}^x=\begin{cases}0 & x<0\\ a_{ij}^x & x\geq 0\end{cases}$ for $i,j=1,2,\ldots,m$ and in the general case we have $\bar{d}_i=\max_{j=1,2,\ldots,m}\{d_{ij}\}$ for $i=1,2,\ldots,m$. Again, in the case of no active constraints the optimal solution is the same as (12a)

$$\boldsymbol{u}(n)=\boldsymbol{\Xi}(\boldsymbol{w}(n)-\boldsymbol{b}(n))-\boldsymbol{\Psi}\,\bar{\boldsymbol{x}}(n) \tag{53}$$

where

$$\begin{aligned}&\boldsymbol{\Psi}=[\boldsymbol{\Psi}_1\,\boldsymbol{\Psi}_2\cdots\boldsymbol{\Psi}_m],\quad \boldsymbol{\Psi}_x=[\boldsymbol{\Psi}_{x_{ij}}],\quad \boldsymbol{\Xi}=[\xi_{ij}]\quad i,j=1,2,\ldots,m\\ &\bar{\boldsymbol{x}}(n)=[v_{11}(n+d_{11})\ldots v_{1m}(n+d_{1m})\ldots\ldots v_{m1}(n+d_{m1})\ldots v_{mm}(n+d_{mm})]^T\end{aligned} \tag{54}$$

where $\boldsymbol{\Xi}$ and $\boldsymbol{\Psi}$ are chosen properly from the elements of $\bar{\boldsymbol{\Xi}}$ and $\bar{\boldsymbol{\Psi}}$ in (10). As the predicted values of $v_{ij}(n+d_{ij})$ for $i,j=1,2,\ldots,m$ are required, mathematical manipulations are performed to show that

$$v_{ij}(n+d_{ij})=a_{ij}^{d_{ij}}v_{ij}(n)+(1-a_{ij})k_{ij}\sum_{s=1}^{d_{ij}}a_{ij}^{s-1}u_j(n-d_{ij}+s-1)\quad i,j=1,2,\ldots,m \tag{55}$$

**Closed loop analysis.** Let $\boldsymbol{b}(n)=\mathbf{0}$, using (48)–(49b) and (1b), it can be shown that

$$\bar{\boldsymbol{x}}(n)=\mathbf{H}(z)\mathbf{G}(z)^{-1}\boldsymbol{y}(n) \tag{56a}$$

where

$$\mathbf{H}(z)=[\mathbf{H}_1(z)^T\,\mathbf{H}_2(z)^T\ldots\mathbf{H}_m(z)^T]^T,\ \mathbf{H}_i(z)=\mathrm{diag}\{\bar{g}_{i1}(z),\bar{g}_{i2}(z),\ldots,\bar{g}_{im}(z)\}\quad i=1,2,\ldots,m \tag{56b}$$

and also,

$$[\mathbf{I}+\boldsymbol{\Psi}\,\mathbf{H}(z)]\mathbf{G}(z)^{-1}\boldsymbol{y}(n)=\boldsymbol{\Xi}\,\boldsymbol{w}(n) \tag{57}$$

where

$$\boldsymbol{\Xi}=[\mathbf{I}+\boldsymbol{\Psi}\,\mathbf{H}(1)]\mathbf{K}^{-1} \tag{58}$$

Hence, the closed loop relation can be obtained as

$$\boldsymbol{y}_p(n)=\mathbf{G}_{cl}(z)\boldsymbol{w}(n) \tag{59a}$$

where

$$\mathbf{G}_{cl}(z) = \mathbf{G}(z)[\mathbf{I} + \mathbf{\Psi}\,\mathbf{H}(z)]^{-1}[\mathbf{I} + \mathbf{\Psi}\,\mathbf{H}(1)]\mathbf{K}^{-1} \tag{59b}$$

To further study the closed loop plant behavior let the gain matrix $\mathbf{\Psi}$ be as

$$\mathbf{\Psi}_x = \mathrm{diag}\{\psi_{x_{11}}, \psi_{x_{22}}, \ldots, \psi_{x_{mm}}\} \quad x = 1, 2, \ldots, m \tag{60}$$

Then, the following characteristic polynomials are derived

$$\begin{aligned}
&\prod_{i=1}^{m}(z - a_{i1}) + \sum_{x=1}^{m}\psi_{x_{11}}k_{x1}(1 - a_{x1}) \prod_{\substack{j=1\\ j\neq x}}^{m}(z - a_{j1}) = 0\\
&\prod_{i=1}^{m}(z - a_{i2}) + \sum_{x=1}^{m}\psi_{x_{22}}k_{x2}(1 - a_{x2}) \prod_{\substack{j=1\\ j\neq x}}^{m}(z - a_{j2}) = 0\\
&\qquad\qquad\vdots\\
&\prod_{i=1}^{m}(z - a_{im}) + \sum_{x=1}^{m}\psi_{x_{mm}}k_{xm}(1 - a_{xm}) \prod_{\substack{j=1\\ j\neq x}}^{m}(z - a_{jm}) = 0
\end{aligned} \tag{61}$$

And it is now possible to place $m$ poles in the desired locations by choosing the $m$ gain variables of the gain matrix in each equation of (61).

### 5.2. Tuning strategy for desired performance

In this section, assuming control horizon of one and inactive constraints, the tuning formulas are derived. We have

$$[\mathbf{\Psi}_1\ \mathbf{\Psi}_2 \ldots \mathbf{\Psi}_m] = (\mathbf{R} + \mathbf{S}^T\mathbf{Q}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}} \tag{62}$$

Eqs. (62) and (52) give

$$\begin{aligned}
(\mathbf{R} + \mathbf{S}^T\mathbf{Q}\mathbf{S})\mathbf{\Psi}_1 &= \mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}}_1\\
(\mathbf{R} + \mathbf{S}^T\mathbf{Q}\mathbf{S})\mathbf{\Psi}_2 &= \mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}}_2\\
&\vdots\\
(\mathbf{R} + \mathbf{S}^T\mathbf{Q}\mathbf{S})\mathbf{\Psi}_m &= \mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}}_m
\end{aligned} \tag{63}$$

**Theorem 4.** In the case of no active constraint and no model mismatch, the optimization problem (8a) for the FOPDT model (1b) leads to the closed loop transfer function matrix (59b) with the desired achievable gain matrix $\mathbf{\Psi}_d = [\mathbf{\Psi}_{d1}\ \mathbf{\Psi}_{d2} \cdots \mathbf{\Psi}_{dm}]$ that satisfies the following inequalities

$$\begin{aligned}
&\mathbf{Q} > 0\\
&\mathbf{\Psi}_{di} < (\mathbf{S}^T\mathbf{Q}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}}_i \quad i = 1, 2, \ldots, m
\end{aligned} \tag{64}$$

and the feasibility conditions of this desired gain matrix is the intersection of (64) and the stability conditions of the closed loop plant (59a). By selecting $\mathbf{R}$ and $\mathbf{Q}$ as the tuning parameters, the tuning relations for achieving this desired gain matrix is

$$\begin{aligned}
&\mathbf{S}^T\mathbf{Q}[\bar{\mathbf{F}}_1\,\mathbf{\Psi}_{d1}^{-1}\mathbf{\Psi}_{di} - \bar{\mathbf{F}}_i] = 0 \quad i = 2, 3, \ldots, m\\
&\mathbf{R} = \mathbf{S}^T\mathbf{Q}[\bar{\mathbf{F}}_1\mathbf{\Psi}_{d1}^{-1} - \mathbf{S}]
\end{aligned} \tag{65}$$

**Proof:** The proof follows from Eq. (63) and the positivity of $\mathbf{R}$ and $\mathbf{Q}$.

Note that, obtaining the analytical solution of (65) is not straight forward. However, it is possible to reformulate it as an optimization problem

$$\begin{aligned}
&\min_{\mathbf{Q}} \sum_{i=2}^{m} \left\|\mathbf{S}^T\mathbf{Q}[\bar{\mathbf{F}}_1\mathbf{\Psi}_{d1}^{-1}\mathbf{\Psi}_{di} - \bar{\mathbf{F}}_i]\right\|\\
&s.t. \quad (\mathbf{S}^T\mathbf{Q}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{Q}\bar{\mathbf{F}}_i - \mathbf{\Psi}_{di} > 0 \quad i = 1, 2, \ldots, m\\
&\qquad\qquad \mathbf{Q} > 0
\end{aligned} \tag{66}$$

and for $\mathbf{R}$ we have the exact solution as

$$\mathbf{R} = \mathbf{S}^T\mathbf{Q}[\bar{\mathbf{F}}_1\mathbf{\Psi}_{d1}^{-1} - \mathbf{S}] \tag{67}$$

## 6. Simulation examples

In this section, simulation results are used to verify the proposed tuning strategies and theorems.

### 6.1. The multivariable pH neutralization process

Complexity of the pH process [5] increases in the multivariable case, where the acid and base streams are the control signals, the pH of the outlet stream and the vessel level are the outputs. Also, the buffer stream is the disturbance. For the working range of pH = 5.5 up to pH = 7 and the level in the range of 15 up to 20 cm, the following model can be derived for the plant

$$\mathbf{G}_p(s) = \left\{ \begin{bmatrix} k_{11}\dfrac{e^{-\theta_1 s}}{\tau_1 s+1} & k_{12}\dfrac{e^{-\theta_1 s}}{\tau_1 s+1} \\ \dfrac{k_2}{\tau_2 s+1} & \dfrac{k_2}{\tau_2 s+1} \end{bmatrix} \middle| \begin{array}{l} -0.3 \le k_{11} \le -0.65,\ 0.3 \le k_{12} \le 0.65,\ 0.89 \le k_2 \le 1.1 \\ 85 \le \tau_1 \le 95,\ 180 \le \tau_2 \le 210,\ \theta_1 = 30 \end{array} \right\}$$

the sampling time is $T_s = 10$ s and the nominal model parameters are

$$k_{11} = -0.47, k_{12} = 0.47, k_{21} = k_{22} = 0.95, a_1 = 0.8948, a_2 = 0.95, d_1 = 3, d_2 = 0$$

It is desired to have a decoupled closed loop plant with a pH settling time of less than 150 s and level settling time of less than 250 s. According to (24a) and (24b) we choose, $l_{11} = 1.7$ and $l_{22} = 2$. Let $\mathbf{Q} = \mathbf{I}$. The achievability conditions of desired loop gain $\mathbf{L}_d = \begin{bmatrix} 1.7 & 0 \\ 0 & 2 \end{bmatrix}$ leads to $P_1 \le 6$ and $P_2 \le 12$ by employing (29). Selecting $P_1 = P_2 = 5$, (30) gives the tuning parameter as $\mathbf{R} = \begin{bmatrix} 0.1805 & 0.1336 \\ 0.1336 & 0.1805 \end{bmatrix}$.

To show the effectiveness of the proposed tuning method, the set points of Fig. 1 are applied to the system and a white noise with variance of 1e−4 is added to the pH output. Buffer disturbance is used to show the ability of the offset free disturbance rejection. The nominal value of buffer is 0.55 ml/s. A step change is applied at sample time 475 from 0.55 ml/s to 0.275 ml/s. Let the input constraint be $0 \le u_1(n), u_2(n) \le 30$. Fig. 1 shows the results of the tuned predictive pH process control. It is clearly shown that all the control objectives are fulfilled. Also, the response of DMC tuning [10] is presented with the tuning parameters $P = N = 88$, $M = 18$, $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = 2.214\mathbf{I}$. Where $N$ is the model horizon in the DMC algorithm. Fig. 1 shows that, although the tuning method of [10] has a proper response but it leads to large horizons with high computational cost.

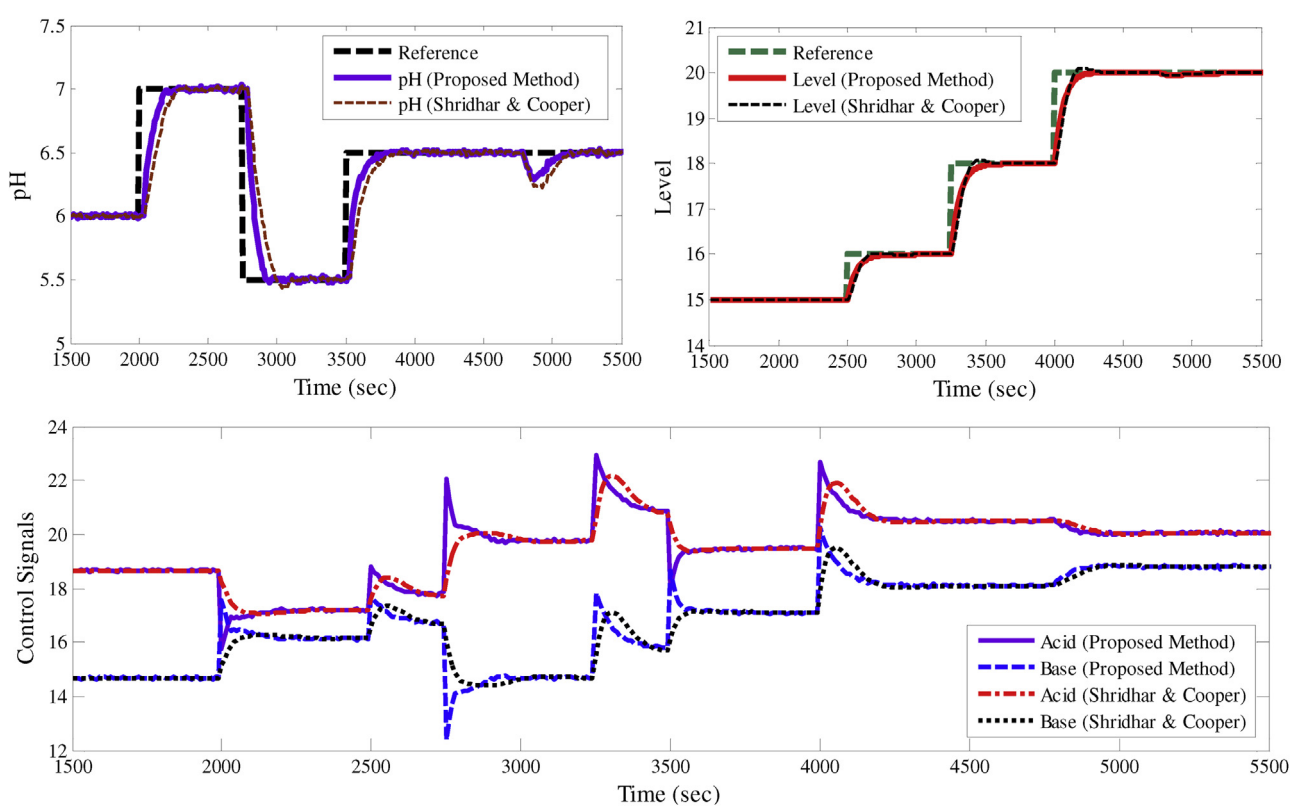


**Fig. 1.** Closed loop responses of the multivariable pH process.

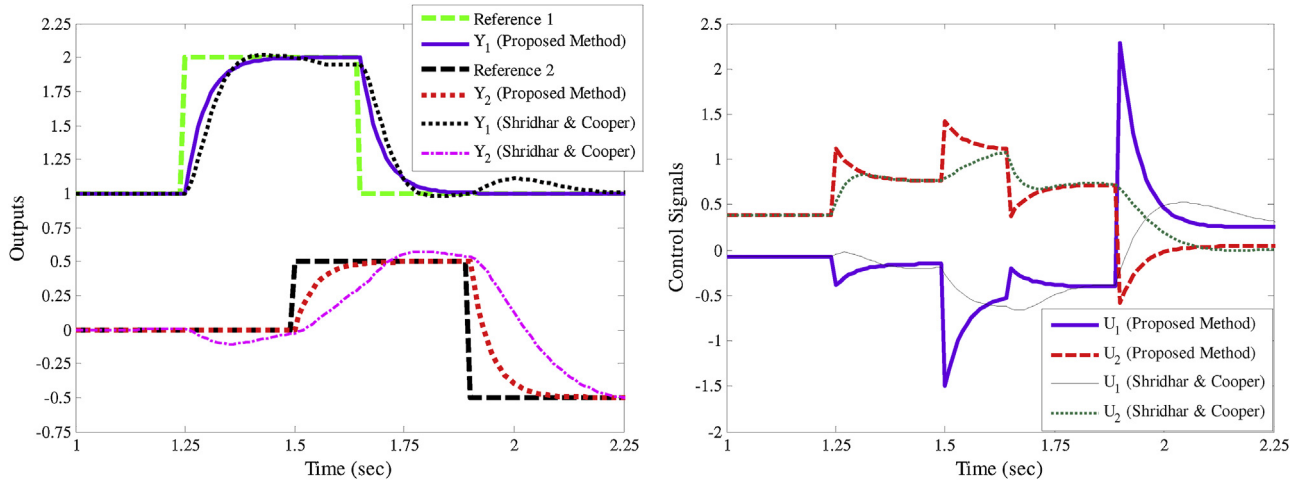


**Fig. 2.** Closed loop responses in example 2.

### 6.2. Second example

Consider the following transfer function [20]

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{16.9}{s+5} & \dfrac{36.12}{s+11} \\ \dfrac{-9.57}{s+5} & \dfrac{-4.175}{s+11} \end{bmatrix}$$

A sampling time of $T_s = 0.01$ s gives

$$\mathbf{K} = \begin{bmatrix} 3.38 & 3.2836 \\ -1.914 & -0.3791 \end{bmatrix}, \quad \mathbf{P}(z) = \mathrm{diag}\left\{ \frac{0.0488}{z-0.9512}, \frac{0.1042}{z-0.8958} \right\}, \quad \mathbf{D}(z) = \mathbf{I}$$

Let the desired closed loop be decoupled as

$$\mathbf{G}_{cl_d}(z) = \mathrm{diag}\left\{ \frac{1-x_1}{z-x_1}, \frac{1-x_2}{z-x_2} \right\}, \; x_1 = x_2 = 0.8$$

according to (45b), we have $\Psi = \mathrm{diag}\{3.1, 0.92\}$ and the feasibility condition is met as given in (46) with $\mathbf{Q} = \mathbf{I}$ and $P_1 = P_2 = 7$. From (47), we have $\mathbf{R} = \begin{bmatrix} 0.708 & 3.479 \\ 1.189 & 6.481 \end{bmatrix}$. Note that, there is no problem with positive definite non symmetric $\mathbf{R}$ and this choice makes the closed loop plant fully decoupled. In Fig. 2, the results of the closed loop responses are shown. It is shown that the desired performance is achieved. In addition, the response of DMC tuning method [10] is presented with the tuning parameters $P = N = 100$, $M = 20$, $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = \mathrm{diag}\{37.72\mathbf{I}, 34.47\mathbf{I}\}$. It is shown that the tuning strategy of [10] leads to inappropriate responses.

### 6.3. The Wood and Berry process [21]

Wood and Berry [21] introduced the following model of a methanol–water distillation column

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}$$

Sampling time of $T_s = 1$ min is selected and the desired closed loop poles are 0.94, 0.9, 0.89, 0.95. Hence, Eq. (61) leads to

$$[\mathbf{\Psi}_1 \; \mathbf{\Psi}_2] = \begin{bmatrix} 0.0036 & 0 & 0.0200 & 0 \\ 0 & -0.0123 & 0 & -0.0274 \end{bmatrix}$$

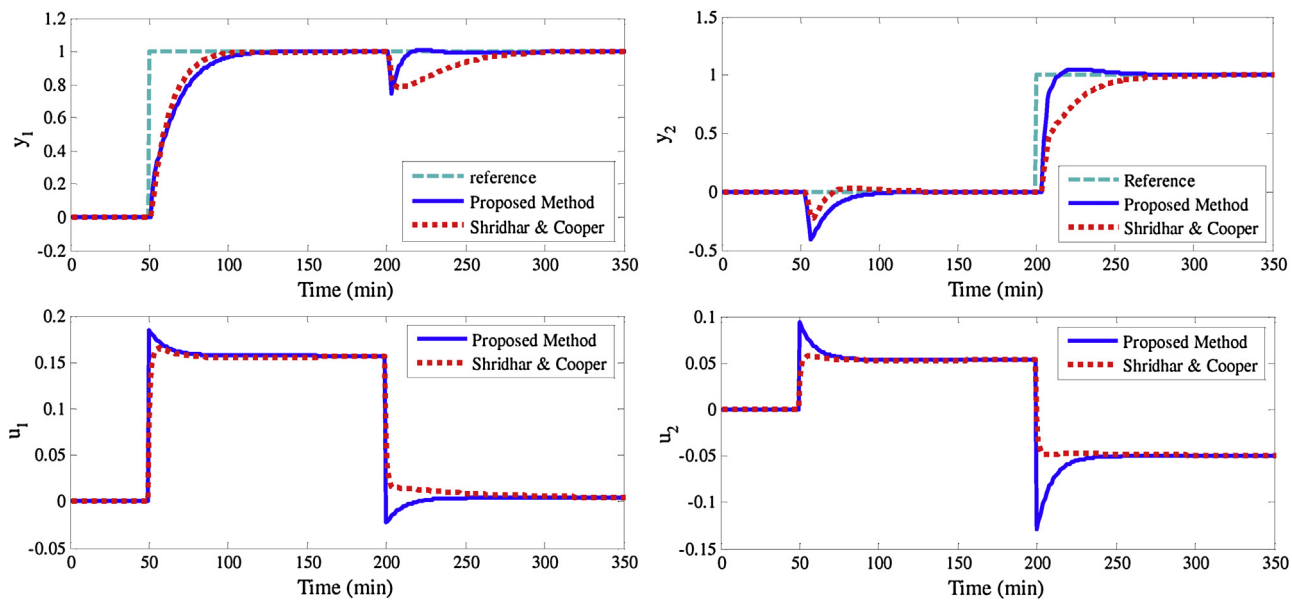


**Fig. 3.** Closed loop responses of Wood and Berry plant.

Now, the optimization problem (66) is solved using (67) and assuming $P = 3$. Hence, we have $\mathbf{Q} = [q_{ij}]$, $q_{ij} = q_{ji}$ $i, j = 1, 2, \ldots, 6$ and $\mathbf{R} = [r_{ij}]$, $r_{ij} = r_{ji}$ $i, j = 1, 2$ and

$$q_{11} = 1.077140,\ q_{12} = -0.175098,\ q_{13} = -0.729254,\ q_{14} = -0.110814,\ q_{15} = -0.552727$$

$$q_{16} = 0.722158,\ q_{22} = 0.619948,\ q_{23} = -0.308832,\ q_{24} = 0.035780,\ q_{25} = 0.6704229$$

$$q_{26} = -0.633336,\ q_{33} = 0.861531,\ q_{34} = 0.356209,\ q_{35} = -0.244493,\ q_{36} = -0.181357$$

$$q_{44} = 1.661151,\ q_{45} = -1.155343,\ q_{46} = -0.349246,\ q_{55} = 1.867961,\ q_{56} = -0.774392$$

$$q_{66} = 1.059596,\ r_{11} = 1.783817,\ r_{12} = -0.846693,\ r_{22} = 0.402034$$

Fig. 3 shows the closed loop responses. The response of DMC tuning [10] is presented with the following parameters

$$T_s = 1 \text{ min},\ P = N = 108, M = 2,\ \mathbf{Q} = \mathbf{I},\ \mathbf{R} = \text{diag}\{\lambda_1\mathbf{I}, \lambda_2\mathbf{I}\},\ \lambda_1 = 70.82,\ \lambda_2 = 235$$

The results show the effectiveness of proposed tuning strategy.

## 7. Conclusions

For multivariable MPC of FOPDT plants, an analytical tuning strategy is developed when the constraints are inactive. The closed loop transfer functions are obtained and the decoupling conditions are derived in special cases. Analytical tuning formulas to achieve the desired performances are derived for control horizon of one. Finally, simulation results are used to show the effectiveness of the proposed tuning approach. Extending the presented results to multivariable unstable and non-square plants, relaxing the limitation on the control horizon, and consideration of multivariable Second Order plus Dead Time (SOPDT) models can be future research directions.

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